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Quasi-Interpolation Functionals on Spline Spaces*

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This paper is concerned with the structure of quasi-interpolation functionals on the space spanned by exponential polynomial splines and their translates. The existence of these functionals is guaranteed by certain conditions which are derived, using the notion of commutators, and shown to be equivalent to some generalization of the Strang–Fix conditions. Characterizations of quasi-interpolation functionals are also formulated, and admissible sets for these functionals are given. Several interpolation schemes are obtained through the quasi-interpolation functionals.

1. INTRODUCTION

Quasi-interpolation functionals play an important role in the construction of approximation formulas using integer-translates of compactly supported functions. We first give a brief review. The following notations will facilitate our discussion.

Let $\alpha=(\alpha_1, ..., \alpha_s)\in Z_+^s$ be a multi-index, $|\alpha|=\sum_{i=1}^s \alpha_i$, and $\forall x\in R^s$ (or $\in \mathbb{C}^s$, where $\mathbb{C}$ is the complex field), let $\|x\|=\max_{1\leqslant i\leqslant s}|x_i|$. Also let $\{\varepsilon^j\}_{j=1}^s$ denote the coordinate basis of R^s. In this paper, we will always assume that a function f is a map from R^s into $\mathbb{C}$, and define, as usual,

$$D^\alpha f(x)=\frac{\partial^{|\alpha|}f}{\partial x_1^{\alpha_1}\,\partial x_2^{\alpha_2}\cdots\partial x_s^{\alpha_s}}(x);$$

$$\sigma_h f(\cdot)=f(\cdot/h),\qquad h>0;$$

$$\tau_y f(\cdot)=f(\cdot+y),\qquad y\in R^s;$$

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and for any $F \subset C(R^s)$, set

$$F^h := \{f : \sigma_h f \in F\}.$$

The Fourier transform of f is given by

$$\hat{f}(z) = \int_{R^s} f(x)\, e^{-ix \cdot z}\, dx, \qquad z \in \mathbb{C}^s.$$

Let $\Omega \subset R^s$, and denote, as usual, the supremum norm on Ω by $\| \; \|_\Omega$. In this paper, we also need the following notations.

$$|f|_{k,\Omega} = \sum_{|\alpha| = k} \|D^\alpha f\|_\Omega;$$

$$\|f\|_{k,\Omega} = \sum_{j=0}^{k} |f|_{j,\Omega};$$

and

$$K_{k,\Omega}(f, h) = \sum_{j=0}^{k} h^j\, |f|_{j,\Omega}.$$

The space of entire functions in $\mathbb{C}^s$ restricted to R^s is denoted by $\mathscr{E}$, and the collection of all polynomials denoted by π. The point evaluation functional δ_x is defined, as usual, by $\delta_x f = f(x)$. Now let ϕ be a compactly supported function. Then space $S(\phi)$ it generates is defined by

$$S(\phi) = \operatorname{span}\{\phi(\cdot - \alpha) : \alpha \in Z^s\}.$$

DEFINITION 1.1 (Cf. [2, 8, 10]). An operator $Q: C^d \mapsto S(\phi)$, where $C^d = C^d(R^s)$, is called a quasi-interpolation operator of order n for ϕ, if

1. Q is a local linear operator, where locality means that

$$\operatorname{supp}(\delta_x Q) \subset C(x, r)$$

for some $r > 0$ independent of x. Here, $C(x, r) = \{y : \|y - x\| \leqslant r/2\}$;

2. Q is locally bounded in the sense that $\exists c > 0$ such that $\forall f \in C^d$ and $x \in R^s$,

$$|(\delta_x Q) f| \leqslant c \|f\|_{d, C(x, r)},$$

where c is a constant independent of x and f; and

3. $Qp = p$, $\forall p \in \pi_{n-1}$, where π_{n-1} is a collection of all polynomials of degree at most $n - 1$.

Now let $Q_h = \sigma_h Q \sigma_{1/h}$. The approximation power of the operator Q_h is described as follows (cf. [1–3, 5, 6], etc.).

THEOREM A. *Suppose that $\Omega \subset R^s$ is an open and convex set, and $A \subset \Omega$ is a compact set. Let $Q: C^d \mapsto S(\phi)$ be a quasi-interpolation operator of order n for ϕ, and*

$$\rho = \rho_{n,d} := \max(n, d). \tag{1.1}$$

Then $\exists c_\Omega > 0$ such that $\forall f \in C^\rho$,

$$\|f - Q_h f\|_A \leqslant c_\Omega \|f\|_{\rho,\Omega} h^n.$$

A natural approach to constructing quasi-interpolation operators is via quasi-interpolation functionals ([9, 11, 14], etc.). We will use the following definition (cf. [10]).

DEFINITION 1.2. A linear functional λ on C^d is called a quasi-interpolation functional order of n for ϕ if

(1) it is local, i.e., supp $\lambda \subset C(0, r)$ for some $r > 0$;

(2) it is bounded, i.e., $\exists c > 0$ such that $\forall f \in C^d$, $|\lambda f| \leqslant c \|f\|_{d, C(0,r)}$; and

(3) for any $p \in \pi_{n-1}$,

$$p(x) = \sum_{j \in Z^s} \lambda p(\cdot + j) \phi(x - j).$$

It is easy to see that a quasi-interpolation functional λ always generates a quasi-interpolation operator Q. Also, sufficient conditions for the existence of quasi-interpolation functionals are available in the literature. In particular, the following set of conditions is usually attributed to Strang and Fix (cf. [6, 21]),

$$\hat{\phi}(0) \neq 0,$$

$$D^\alpha \hat{\phi}(2\pi j) = 0, \qquad \forall j \in Z^s \backslash \{0\}, \quad |\alpha| < n.$$

Various approaches to constructing quasi-interpolation functionals have been studied (cf. [2, 3, 5, 9, 10, 14]).

Observe that in fact the operator Q_h realizes the approximation order of the family of spaces

$$S_h(\phi_h) = \text{span}\{\phi_h(\cdot - \alpha) : \alpha \in hZ^s\}$$

with $\phi_h = \sigma_h \phi$, where the approximation order describes a "distance" between the space C^d and $S_h(\phi_h)$. Since the common subspace contained in all the spaces $S_h(\phi_h)$, $h > 0$, must be a subspace of the space of all polynomials, the Strang–Fix condition is important for describing the approximating ability of $S_h(\phi_h)$ here. However, if we are not restricted by the definition $\phi_h = \sigma_h \phi$ but instead, allow ϕ_h, $h > 0$, to be a family of functions then the situation will be quite different. A typical example is the exponential box spline setting, where the original version of Strang–Fix condition is not applicable. Discussion of exponential box splines can be found in [4, 7, 15–17, 19, 20], etc. The essential questions are as follows: Which space is contained in all the spaces $S(\phi_h)$, $h > 0$, and What approximation order does this family of spaces achieve? We will discuss these problems in a more general setting.

DEFINITION 1.3. Suppose that $0 < h \leqslant 1$, and k is a certain non-negative integer. Then $\Phi = \{\phi_h\}_{0 < h \leqslant 1} \subset C^k(R^s)$ is called a G-family if there exists a positive number r such that $\operatorname{supp} \phi_h \subset C(0, hr)$ and

$$\|\Phi\| := \sup_{h > 0} \|\phi_h\| < +\infty.$$

Furthermore, if, in addition,

$$\inf_{h > 0} |\hat{\phi}_h(0)|\, h^{-s} > 0$$

we call Φ a regular G-family.

Later if no confusion would arise, we will omit h when $h = 1$. In this paper we mainly consider the local approximation properties of the family of spaces $S_h(\phi_h)$ generated by a G-family Φ as well as quasi-interpolation functionals for this kind of approximation. In Section 2, we shall give a precise definition of quasi-interpolation operators of order n for Φ as local linear operators $Q_h : C^d \mapsto S_h(\phi_h)$ which enable us to achieve the local approximation order n for any $f \in C^\rho$ (cf. (1.1) for the definition of ρ). It will be seen that the definition of quasi-interpolation functinals for Φ can then be formulated quite naturally. In Section 3, using the notion of commutators (cf. [8, Chap. 8]), we obtain conditions that guarantee the existence of quasi-interpolation functionals for Φ. We will also prove that these conditions are equivalent to certain conditions on the Fourier transform of ϕ_h, generalizing the Strang-Fix conditions (cf. [17]). Characterizations of the quasi-interpolation functionals for a G-family are given in Section 4. In the final section, we deal with admissible sets for quasi-interpolation functionals, which were first discussed in [10]. We will also point out the relation between the dual spaces for $S_h(\phi_h) \cap \mathscr{E}$ and admissible sets. As an application, we study the boundedness properties of the dual basis for the space of integer-translates of an exponential box spline.

2. Quasi-Interpolation Functionals for a G-Family

For $h > 0$, a subspace of $C(R^s)$ is called an h-translation invariant space if

$$f \in H \Rightarrow \tau_{\pm h\varepsilon^j} f \in H, \qquad j = 1, 2, \ldots s;$$

and H is called a translation invariant space (TIS) if it is h-translation invariant for any $h > 0$.

Let Φ be a G-family and $H(\phi_h) := S_h(\phi_h) \cap \mathscr{E}$. If $\exists h_0 > 0$ such that $\forall h$, with $h_0 \geqslant h > 0$, $H(\phi_h) = H(\phi_{h_0})$, then $H(\phi_{h_0})$ is a finite dimensional TIS, which has a fairly simple structure as described in the following theorem.

Theorem B [4]. *If H is a finite dimensional TIS, then $H \subset E$, where E is the space spanned by all exponential polynomials.*

Later when we consider an arbitrary TIS, we always assume that it is finite dimensional. Recall that a TIS can be decomposed into a direct sum of several translation invariant subspaces ([4, 7], etc.). Let

$$\Theta = \{\theta \in \mathbb{C} : \exp(\theta \cdot x) \in H\}$$

be the set of eigenvalues of H. Then

$$H = \bigoplus_{\theta \in \Theta} H_\theta,$$

where $H_\theta = \{e^{\theta \cdot x} p : p \in P_\theta\}$, with $P_\theta \subset \pi$, is a TIS.

Setting $\dim H_\theta = n_\theta$, $d_\theta = \deg H_\theta := \deg P_\theta := \max_{p \in P_\theta} \deg p$, we call θ a simple eigenvalue if $\deg H_\theta = 0$; otherwise, it is called a multiple one. It is obvious that $\forall \theta \in \Theta$ and l, with $d_\theta \geqslant l \geqslant 0$, $P_\theta^l = P_\theta \cap \pi^l$ is a TIS. Hence, we can choose a basis $\{p_j^\theta\}_{j=1}^{n_\theta}$ of P_θ such that for any l, $\{p_j^\theta\}_{j=1}^{n_\theta} \cap \pi_l$ is also a basis P_θ^l. Then we shall call $\{e^{\theta \cdot x} p_j^\theta\}_{j=0}^{n_\theta}$ a canonical basis of H_θ and

$$\bigcup_{\theta \in \Theta} \{e^{\theta \cdot x} p_j^\theta\}_{j=1}^{n_\theta}$$

a canonical basis of H.

Definition 2.1. Let k be a non-negative integer, Λ an n-dimensional space of complex linear functionals defined on $C^k(R^s)$, and $\{\lambda_j\}_{j=1}^n$ a basis of Λ. A space $H \subset C^k(R^s)$ is called Λ-poised if for any $\{y_j\}_{j=1}^n \in \mathbb{C}^n$, there exists a function $g \in H$ such that

$$\lambda_j(g) = y_j, \qquad j = 1, ..., n.$$

We will specify this property of H by writing $H \in I(\Lambda)$. Additionally, if $\dim \Lambda = \dim H$, we write $H \in I_d(\Lambda)$. Later we will use the notation

$$F^* = \{\delta_0 p(D) : p \in F\},$$

where F is any subspace of π.

DEFINITION 2.2 Let H be a TIS. A family of linear operators $\{Q_h\}_{h>0} : C^d \mapsto S_h(\phi_h)$ is called a family of H-reproducing operators for a G-family Φ if, for any $h > 0$,

(1) there exists a positive number r, independent of h and x, such that

$$\operatorname{supp}(\delta_x Q_h) \subset C(x, hr); \tag{2.1}$$

(2) $Q_h f = f$ for any $f \in H$.

If $H \in I(\pi_{n-1}^*)$, then we say that Q_h is of order n. Furthermore, if $\{Q_h\}_{h>0}$ satisfies the uniform boundedness condition

$$|Q_h f(x)| \leqslant c K_{d, C(x, hr)}(f, h), \qquad f \in C^d, \tag{2.2}$$

where c is a constant independent of f and h, then we call $\{Q_h\}_{h>0}$ a family of quasi interpolation operators.

For convenience, we will also say that "Q_h is a quasi-interpolation operator." The following theorem explains the meaning of the above definition. It can be proved essentially in the same way as in [16, 19].

THEOREM 2.1. *Suppose that $\Omega \subset R^s$ is open, $A \subset \Omega$ a compact set, $Q_h : C^d \mapsto S_h(\phi_h)$ a quasi-interpolation operator of order n for a G-family Φ, and $\rho = \rho_{n,d}$ as defined in* (1.1). *Then $\forall f \in C^\rho$,*

$$\|f - Q_h f\|_A \leqslant c_\Omega \|f\|_{\rho, \Omega} h^n,$$

where c_Ω is a constant independent on f and h.

We remark that upper bound estimates for the approximation from $S_h(\phi_h)$ can be found in [19]. Let us now introduce the concept of quasi-interpolation functionals for a G-family Φ.

DEFINITION 2.3. Let H be a TIS. A linear functional λ_h on $C^d(R^s)$ is called an H-reproducing functional for a G-family Φ, if

(1) $\exists r > 0$, independent of h, such that

$$\operatorname{supp} \lambda_h \subset C(0, hr); \tag{2.3}$$

(2) for any $f \in H$,

$$f(x) = \sum_{j \in hZ^s} \lambda_h f(\cdot + j)\, \phi_h(x - j). \tag{2.4}$$

We also say that λ_h is of order n, if $H \in I(\pi^*_{n-1})$. Furthermore, if

(3) λ_h is uniformly K-bounded with respect to h; i.e.,

$$|\lambda_h f| \leqslant c K_{d, C(0, hr)}(f, h), \qquad \forall f \in C^d, \tag{2.5}$$

where c is a constant independent on h and f, then we call λ_h a quasi-interpolation functional.

It is obvious that any quasi-interpolation functional λ_h generates a quasi-interpolation operator Q_h.

3. COMMUTATORS FOR A G-FAMILY

Following [8, Chap. 8], we introduce the definition of the h-commutator of a compactly supported generator $\psi \in C_0(R^s)$ (cf. also [2, 7, 12, 13, 22]).

DEFINITION 3.1. The h-commutator of a compactly supported function $\psi \in C_0(R^s)$ is an operator on $C(R^s)$ defined by

$$[\psi \mid f]_h(x) = \sum_{j \in hZ^s} \psi(x - j) f(j) - \sum_{j \in hZ^s} f(x - j)\, \psi(j), \qquad f \in C(R^s).$$

Using the notation

$$T^h_\phi \eta(x) = \sum_{j \in hZ^s} \phi(x - j)\, \eta(j),$$

where $\phi \in C_0(R^s)$ and $\eta \in C(R^s)$ (or $\phi \in C(R^s)$ and $\eta \in C_0(R^s)$), we can rewrite the above as

$$[\psi \mid f]_h(x) = T^h_\psi f(x) - T^h_f \psi(x).$$

In the following, we shall see that h-commutators play an important role in the investigation of the properties of a G-family. To study these operators we first introduce certain differential operators. Let $\mathcal{O}$ be the space of functions analytic at the origin. For $f \in \mathcal{O}$, we define an operator $f(D): \pi \mapsto \pi$ by

$$f(D)\,p(x) = \sum_{\alpha \geqslant 0} \frac{1}{\alpha!} D^\alpha f(0)\, D^\alpha p(x).$$

If $f \in \mathcal{E}$ (or $\in E$, or $\in \pi$), then the operator $f(D)$ can be extended to E ($\mathcal{E}$, or $\mathcal{O}$, respectively). Observe that for any $g \in \mathcal{E}$, we have $e^{z \cdot D} g(\cdot) = g(z + \cdot)$, and thus the operator $e^{z \cdot D}$ is identified with τ_z. We extend $e^{z \cdot D}$ to $C(R^s)$ by using this formula. More generally, if $f = \sum_{z \in A} e^{z \cdot (\cdot)} p_z$, where A is finite subset of $\mathbb{C}$ and $p_z \in \pi$, then $f(D)$ is well defined on $\mathcal{O}(A) = \{f : f(\cdot - z) \in \mathcal{O}, z \in A\}$. On the other hand, if $d = \max_{z \in A} \deg p_z$, then $f(D)$ can also be extended to $C^d(R^s)$ as mentioned above.

Now if f is fixed in some space ($\mathcal{O}, \mathcal{E}, E$, or π), then the functional $\delta_x f(D)$ is well defined in the corresponding space just mentioned. We set

$$\langle f, g \rangle := (f(-iD)g)(0) = (f(-iD)g)(x)|_{x=0}.$$

Later when we consider $\langle f, g \rangle$, we always assume that f and g are in suitable spaces so that $\langle f, g \rangle$ is well defined.

LEMMA 3.1. *Let* $g(x) = e^{\theta \cdot x} p(x)$, $p \in \pi$ *and* $\phi \in C_0(R^s)$, *such that* $\{(g(iD)\,\hat{\phi})(2\pi j)\}_{j \in Z^s} \in l^1(Z^s)$. *Then*

$$[g \mid \phi](x) = \sum_{j \in Z^s} [(g(x - iD)\hat{\phi})(2\pi j)](e^{2\pi i j \cdot x} - 1). \tag{3.1}$$

Proof. It follows from [8, Theorem 8.2] that (3.1) holds for $\theta = 0$. Now if $\theta \neq 0$, by setting $\psi(x) = e^{-\theta \cdot x} \phi(x)$, we have $\psi \in C_0(R^s)$ and $\{(p(-iD)\,\hat{\psi})(2\pi j)\} \in l^1(Z^s)$. Then

$$\begin{aligned}[g \mid \phi](x) &= e^{\theta \cdot x}[p \mid \psi](x) \\ &= e^{\theta \cdot x} \sum_{j \in Z^s} [(p(x - iD)\hat{\psi})(2\pi j)](e^{2\pi i j \cdot x} - 1) \\ &= \sum_{j \in Z^s} [(g(x - iD)\hat{\phi})(2\pi j)](e^{2\pi i j \cdot x} - 1).\end{aligned}$$

THEOREM 3.1. *Let H be a TIS with eigenvalue set Θ. Then the following statements are equivalent.*

(a) *For any* $g \in H_\theta$, $\theta \in \Theta$,

$$g(-ihD)\,\hat{\phi}_h(\cdot/h)|_{2\pi j} = 0, \qquad \forall j \in Z^s \backslash \{0\}. \tag{3.2}$$

(b) *For any* $g \in H_\theta$, $\theta \in \Theta$,

$$[g \mid \phi_h]_h = 0. \tag{3.3}$$

(c) *For any* $\theta \in \Theta$,

$$T^h_{\phi_h}(H_\theta) \subset H_\theta. \tag{3.4}$$

Proof. First we observe that

$$[g \mid \phi_h]_h\,(hx) = [\sigma_{1/h}\, g \mid \sigma_{1/h}\phi_h](x). \tag{3.5}$$

Hence, without loss of generality, we may assume that $h = 1$.

(i) (a)$\Leftrightarrow$(b).

If (a) holds, then (3.2)$\Rightarrow \{(g(-iD)\,\hat{\phi}(2\pi j)\} \in l^1(Z^s)$. By Lemma 3.1, recalling that H_θ is a TIS, we have $[g \mid \phi] = 0$, i.e., (b) holds.

If (b) holds, then since $g \in H_\theta \Rightarrow g_x(\cdot) := g(\cdot - x) \in H_\theta$, we have, by (3.3), $[g_x \mid \phi] = 0$, $\forall x \in R^s$, i.e.,

$$\sum_{j \in Z^s} g(j-x)\,\phi(x-j) - \sum_{j \in Z^s} g(-j)\,\phi(j) = 0. \tag{3.6}$$

Now let u be an arbitrary rapidly decreasing function. Then

$$\psi(t) = \int g(t-x)\,\phi(x-t)\,u(x)\,dx$$

is also rapidly decreasing. By (3.6) and applying the Poisson summation formula to ψ, we have

$$\sum_{j \in Z^s} g(-j)\,\phi(j)\,\hat{u}(0) = \int \sum_{j \in Z^s} g(j-x)\,\phi(x-j)\,u(x)\,dx$$
$$= \sum_{j \in Z^s} [(g(-iD)\,\hat{\phi})(2\pi j)]\,\hat{u}(2\pi j).$$

Since u is arbitrary, assertion (3.2) holds.

(ii) (b)$\Leftrightarrow$(c).

That (b)$\Rightarrow$(c) is trivial. On the other hand, if (c) holds, then $[g \mid \phi](x) \in H_\theta$, $\forall g \in H_\theta$. Since any nonzero function in H_θ cannot vanish on Z^s, it follows that the condition $[g \mid \phi](j)=0$, $\forall j \in Z^s$, together with the condition $[g \mid \phi] \in H_\theta$ implies $[g \mid \phi]=0$. Hence, (c) $\Rightarrow$ (b) also holds.

COROLLARY 3.1. *Let H be a TIS. If $[g \mid \phi_h]_h=0$, $\forall g \in H$, then*

$$(\sigma_{1/h} T^h_{\phi h} g)(x) = \langle \sigma_{1/h} g(x+\cdot), \hat{\psi}_h \rangle, \qquad \forall x \in R^s, \quad g \in H, \tag{3.7}$$

where $\psi_h = \sigma_{1/h}\phi_h$.

Proof. That (3.2) holds is a consequence of $[g \mid \phi_h]_h=0$, $\forall g \in H$. This in turn yields that

$$(\sigma_{1/h} g)(-iD)\, \hat{\phi}_h\left(\frac{\cdot}{h}\right)\Bigg|_{\cdot\, = 2\pi j} = 0, \qquad \forall j \in Z^s \backslash \{0\},$$

which, in view of the fact that H is a TIS, implies that

$$(\sigma_{1/h} g)(x-iD)\, \hat{\phi}_h\left(\frac{\cdot}{h}\right)\Bigg|_{\cdot\, = 2\pi j} = 0, \qquad \forall j \in Z^s \backslash \{0\}.$$

Applying the Poisson summation formula to the function $\phi_h(\cdot/h)\, g((x-\cdot)/h)$, we obtain

$$\begin{aligned}\sum_{j \in hZ^s} g(hx-j)\, \phi_h(j) &= \sum_{j \in Z^s} (\sigma_{1/h} g)(x-j)(\sigma_{1/h}\phi_h)(j) \\ &= (\sigma_{1/h} g)(x-iD)\widehat{(\sigma_{1/h}\phi_h)}(0) \\ &= \langle \sigma_{1/h} g(x+\cdot), \hat{\psi}_h \rangle.\end{aligned}$$

On the other hand, by applying $[g \mid \phi_h]_h=0$ we have

$$(\sigma_{1/h} T^h_{\phi_h} g)(x) = \sum_{j \in hZ^s} \phi_h(hx-j)\, g(j) = \sum_{j \in hZ^s} g(hx-j)\, \phi_h(j).$$

Hence, we obtain (3.7).

The commutator property (3.3) is usually called the set of H_θ-vanishing conditions of the h-commutator for ϕ_h. We remark that when H_θ is a space of all algebraic polynomials, these equivalent statements are well known (cf. [8]). We are now ready to establish the following main result of this section.

THEOREM 3.2. *Let Φ be a regular G-family and H a TIS with eigenvalue set Θ. Then for any sufficiently small $h>0$, the following statements are equivalent*:

(a) *For any* $g \in H_\theta, \theta \in \Theta$,

$$g(-ihD)\,\hat{\phi}_h\left(\frac{\cdot}{h}\right)\Big|_{2\pi j} = 0, \qquad \forall j \in Z^s \setminus \{0\}.$$

(b) *For any* $g \in H_\theta, \theta \in \Theta$,

$$[g \mid \phi_h]_h = 0.$$

(c) $T^h_{\phi_h}$ *is an automorphism on* H_θ, $\forall \theta \in \Theta$.

(d) *There exists a quasi-interpolation functional* λ_h *for* H *and* Φ.

Proof. By Theorem 3.1, we only need to prove (b) $\Rightarrow$ (c) and (c) $\Leftrightarrow$ (d).

(i) (b) $\Rightarrow$ (c).

It has been shown in Theorem 3.1 that (b) implies

$$\sum_{j \in hZ^s} e^{-\theta \cdot j}\phi_h(j) = \hat{\psi}_h(-ih\theta), \qquad \forall \theta \in \Theta, \tag{3.8}$$

where $\psi_h = \sigma_{1/h}\phi_h$. Now, that the regularity of Φ implies $\inf_{h>0} |\hat{\psi}_h(0)| > 0$. Hence, there exits a constant $\rho > 0$ independent of h, such that $\hat{\psi}_h(z) \neq 0, \forall z \in C(0, \rho)$. Since Θ is a finite set, there exist some $h_0 > 0$ and $c > 0$, such that $\forall h$ with $h_0 \geqslant h > 0$,

$$|\hat{\psi}_h(-ih\theta)| \geqslant c > 0. \tag{3.9}$$

Since $\deg(g(x-j) - g(x)\,e^{-\theta \cdot j}) < \deg g$, $\forall g \in H_\theta$, we have

$$\begin{aligned} & g(x) - \frac{1}{\hat{\psi}_h(-ih\theta)}\, T^h_{\phi_h} g(x) \\ &= \frac{1}{\hat{\psi}^h(-ih\theta)}\left(\sum_{j \in hZ^s} g(x)\, e^{-\theta \cdot j}\phi_h(j) - T^h_g\,\phi_h(x)\right) \\ &= \frac{1}{\hat{\psi}_h(-ih\theta)} \sum_{j \in hZ^s} (g(x)\, e^{-\theta \cdot j} - g(x-j))\,\phi_h(j) \\ &\Rightarrow \deg\left(g(x) - \frac{1}{\hat{\psi}_h(-ih\theta)}\, T^h_{\phi_h} g(x)\right) < \deg g(x). \end{aligned} \tag{3.10}$$

This together with the fact that $T^h_{\phi_h}(H_\theta) \subset H_\theta$ implies (c).

(ii) (c) $\Rightarrow$ (d).

We will use the Neumann series approach introduced in [11]. (Also, cf. [7, 16, 19], etc.). Write

$$\rho_{h\theta} = (\hat{\psi}(-ih\theta))^{-1}$$

and set

$$q_h(x) = 1 - \prod_{\theta \in \Theta} (1 - \rho_{h\theta} x)^{d_\theta + 1}.$$

By (3.10), we have

$$(I - \rho_{h\theta} T^h_{\phi_h})^{d_\theta + 1} g = 0, \qquad \forall g \in H_\theta,$$

where I is the identity operator and $d_\theta = \deg H_\theta$. This implies that $q(T^h_{\phi_h}) g = g$, $\forall g \in H$.

Denoting the cardinality of $C(0, r) \cap Z^s$ by J, we obtain

$$|T^h_{\phi_h} f(x)| \leqslant 2J \, \|f\|_{0, C(x, 2hr)} \, \|\phi_h\|, \tag{3.11}$$

and that supp $\delta_x T^h_{\phi_h} \subset C(0, 2hr)$. Since $q(x)$ is a polynomial that vanishes at the origin, we conclude that $Q_h = q(T^h_{\phi_h})$ is a quasi-interpolation operator for Φ. Now let $b_h(x) = q_h(x)/x$. Then $b_h(x)$ is also a polynomial and

$$b_h(T^h_{\phi_h}) \, T^h_{\phi_h} g = T^h_{\phi_h} b_h(T^h_{\phi_h}) g = g, \qquad \forall g \in H.$$

This means that

$$b_h(T^h_{\phi_h}) = (T^h_{\phi_h})^{-1} \qquad \text{on } H. \tag{3.12}$$

Note that $\tau_k T^h_{\phi_h} = T^h_{\phi_h} \tau_k$, $\forall k \in hZ^s$, so that $\tau_k b_h(T^h_{\phi_h}) = b_h(T^h_{\phi_h}) \tau_k$, $\forall k \in hZ^s$. This allows us to define λ_h by

$$\lambda_h f = b_h(T^h_{\phi_h}) f(0), \tag{3.13}$$

which has the property that

$$\lambda_h f(\cdot + j) = (b_h(T^h_{\phi_h}) f)(j).$$

Hence, λ_h is the quasi-interpolation functional related to Q_h.

(iii) (d) $\Rightarrow$ (c).

This is a direct consequence of the fact that

$$T^h_{\phi_h}(\lambda_h g(\cdot + x)) = g(x), \qquad \forall g \in H_\theta, \quad \theta \in \Theta.$$

4. Characterizations of Quasi-Interpolation Functionals for a G-Family

In Section 3, we obtained conditions that guarantee the existence of quasi-interpolation functionals for a G-family Φ, with one of such functionals given by (3.13). In this section, we shall give several characterizations of these functionals. We always assume that H is a TIS and Φ is a G-family. Also, the H-vanishing conditions of h-commutator for $\phi_h \in \Phi$, namely,

$$[g \mid \phi_h] = 0, \qquad \forall g \in H,$$

will be denoted simply by

$$[H \mid \Phi]_h = 0. \tag{4.1}$$

Recall that the equivalent statements in Theorem 3.2 hold for all sufficiently small $h > 0$. For convenience, when we mention the condition (4.1) later, we always assume that h is so small that these equivalent relations hold.

Let Λ_Q^h denote the set of quasi-interpolation functionals for Φ. The following theorem tell us that $\forall \lambda_h^1, \lambda_h^2 \in \Lambda_Q^h$,

$$\lambda_h^1 g = \lambda_h^2 g, \qquad \forall g \in H.$$

THEOREM 4.1. *If* $[H \mid \Phi]_h = 0$, *then* $\forall \lambda_h \in \Lambda_Q^h, g \in H$,

$$\lambda_h (T_{\phi_h}^h g)(\cdot + x) = g(x). \tag{4.2}$$

In particular,

$$\lambda_h g = (T_{\phi_h}^h)^{-1} g(0) = b_h(T_{\phi_h}^h) g(0). \tag{4.3}$$

Proof. Since $[H \mid \Phi]_h = 0$, $\forall g \in H$, we have

$$\sum_{j \in hZ^s} g(x + y - j)\, \phi_h(j) = \sum_{j \in hZ^s} g(j)\, \phi_h(x + y - j)$$

and, considering that $g_y(x) := g(y + x) \in H$ for any fixed $y \in R^s$, we also have

$$\sum_{j \in hZ^s} g(x + y - j)\, \phi_h(j) = \sum_{j \in hZ^s} g(y + j)\, \phi_h(x - j).$$

Thus,

$$\sum_{j \in hZ^s} g(y + j)\, \phi_h(x - j) = \sum_{j \in hZ^s} g(j)\, \phi_h(y + x - j), \tag{4.4}$$

and it follows that

$$\lambda_h(T^h_{\phi_h} g)(\cdot + x) = \sum_{j \in hZ^s} \lambda_h g(\cdot + j)\, \phi_h(x - j) = g(x).$$

Hence, assertion (4.2) is verified. Also, (4.3) follows by applying (4.2) and (3.12).

Note that the value of $(T^h_{\phi_h})^{-1} g(0)$ is independent of the individual λ_h, and this implies that $\forall \lambda^1_h, \lambda^2_h \in \Lambda^h_Q, g \in H, \lambda^1_h g = \lambda^2_h g$.

Now we shall give another characterization of Λ^h_Q. Since H is finite dimensional, we can characterize Λ^h_Q in terms of a basis of H. Let $\tilde{\eta} = \{\tilde{\eta}_j\}_{j=1}^m$ be a (canonical) basis of H. Using the property (3.10) and the fact that $T^h_{\phi_h}$ is an automorphism on H, we conclude that

$$\eta := \{\eta_j\}_{j=1}^m = \{T^h_{\phi_h} \tilde{\eta}_j\}_{j=1}^m$$

is also a (canonical) basis of H, $\forall \lambda_h \in \Lambda^h_Q$,

$$\lambda_h \eta_j = \tilde{\eta}_j(0), \qquad j = 1, 2, ..., m. \tag{4.5}$$

In the following, we give certain recurrence formulas for determining $\{\tilde{\eta}_j\}_{j=1}^m$ from $\{\eta_j\}_{j=1}^m$.

THEOREM 4.2. *Let H be a TIS with eigenvalue set Θ, Φ a G-family satisfying $[H \mid \Phi]_h = 0$, $\{\eta^\theta_\alpha\}_{\alpha \in A_\theta}$ a canonical basis of H_θ which is assumed to satisfy $\eta^\theta_{\alpha_0} = e^{\theta \cdot (\cdot)}$, $\alpha_0 \in A_\theta$, and $\{\lambda^\theta_\alpha\}_{\alpha \in A_\theta}$* a dual basis of *$\{\eta^\theta_\alpha\}_{\alpha \in A_\theta}$. Then the following recurrence formulas holds for $\tilde{\eta}^\theta_\alpha = (T^h_{\phi_h})^{-1} \eta^\theta_\alpha$,*

$$\begin{cases} \tilde{\eta}^\theta_{\alpha_0} = \rho_{h\theta} \eta^\theta_{\alpha_0} \\ \tilde{\eta}^\theta_\alpha = \rho_{h\theta} \eta^\theta_\alpha - \rho_{h\theta} \sum_{\beta \in B(\alpha)} \tilde{\eta}^\theta_\beta \lambda^\theta_\beta (T^h_{\phi_h} \eta^\theta_\alpha), \end{cases} \tag{4.6}$$

where

$$B(\alpha) = \{\beta \in A_\theta : \deg \tilde{\eta}^\theta_\beta < \deg \tilde{\eta}^\theta_\alpha\}.$$

Proof. That $\tilde{\eta}^\theta_{\alpha_0} = \rho_{h\theta} \eta^\theta_{\alpha_0}$ is trivial. Let

$$\tilde{\eta}^\theta_\alpha = \rho_{h\theta} \eta^\theta_\alpha - \sum_{\beta \in B(\alpha)} a_\beta \tilde{\eta}^\theta_\beta.$$

Then

$$\eta^\theta_\alpha = T^h_{\phi^h} \tilde{\eta}^\theta_\alpha = \rho_{h\theta} (T^h_{\phi^h} \eta^\theta_\alpha) - \sum_{\beta \in B(\alpha)} a_\beta \eta^\theta_\beta.$$

Since $\{\lambda_\alpha^\theta\}_{\alpha\in A_\theta}$ is a dual basis of $\{\eta_\alpha^\theta\}_{\alpha\in A_\theta}$, it follows that

$$a_\beta = -\rho_{h\theta}\lambda_\beta^\theta(T_{\theta h}^h \eta_\alpha^\theta),$$

which is simply (4.6).

Next, we shall establish a theorem characterizing Λ_Q^h via the Fourier transform of ϕ_h. The following lemma is needed for this purpose.

LEMMA 4.1. *The following statements hold*:

(a) $\langle f, g\rangle = \langle g, f\rangle$.

(b) *Let* $g_f(x) = \langle f, g(x+\cdot)\rangle$. *Then* $\langle h, g_f\rangle = \langle hf, g\rangle$.

Proof. Statement (a) is trivial and statement (b) follows from the relationship

$$\begin{aligned}\langle hf, g\rangle &= hf(-iD)g(0) = h(-iD)[f(-iD)g](0) = \langle h, f(-iD)g\rangle \\ &= \langle h, g_f\rangle.\end{aligned}$$

THEOREM 4.3. *Let H be a TIS and Φ a G-family satisfying* $[H\,|\,\Phi]_h = 0$. *Also, let* $\psi_h = \sigma_{1/h}\phi_h$. *Then* $\forall h > 0$, $\hat{\psi}_h^{-1}$ *is analytic in the ball* $C(0, \rho)$, *where* ρ *is a positive constant independent of h. In addition,* $\forall \lambda_h \in \Lambda_Q^h$ *and* $g \in H$,

$$\lambda_h g = \langle \hat{\psi}_h^{-1}, \sigma_{1/h} g\rangle. \tag{4.7}$$

Proof. Since Φ is a G-family, the family $\{\hat{\psi}_h\}_{h>0}$ of entire functions is uniformly bounded, and hence, equicontinuous, on any compact set. Recall that $\hat{\psi}_h(z) \neq 0$, $\forall z \in C(0, \rho)$ for some $\rho > 0$ (cf. the proof (i) of Theorem 3.1). Hence $\hat{\psi}_h^{-1}(z)$ is analytic in $C(0, \rho)$. Since the eigenvalue set of H is finite, it is clear that $\langle \hat{\psi}_h^{-1}, \sigma^{1/h} g\rangle$ is well defined for all $g \in H$ as long as h is sufficiently small. Now we are ready to prove (4.7). $\forall g \in H$, write $\tilde{g} = (T_{\phi_h}^h)^{-1} g$. Since $[H\,|\,\Phi]_h = 0$, it follows by applying Corollary 3.1, that

$$\begin{aligned}\langle \hat{\psi}_h^{-1}, \sigma_{1/h} g\rangle &= \langle \hat{\psi}_h^{-1}, \sigma_{1/h}(T_{h_\phi}^h \tilde{g})\rangle = \langle \hat{\psi}_h^{-1}, (\sigma_{1/h}\tilde{g})_{\hat{\psi}_h}\rangle \\ &= \langle \hat{\psi}_h^{-1}\hat{\psi}_h, \sigma_{1/h}\tilde{g}\rangle = \tilde{g}(0) = (T_{\phi_h}^h)^{-1} g(0).\end{aligned}$$

Thus (4.7) is a consequence of (4.3).

5. ADMISSIBLE SETS FOR A G-FAMILY

In this section we focus our attention on the study of expansions of quasi-interpolation functionals on admissible sets. First, let us briefly explain the main idea.

Let H be a TIS, Φ a G-family such that $[H|\Phi]_h=0$, and Λ_Q^h the set of quasi-interpolation functionals for Φ. Now $\forall\lambda_h\in\Lambda_Q^h$, we define $\mu_h=\lambda_h\sigma_h$. By Definition 2.3, we can see that μ_h satisfies the following conditions:

(a) $\exists\, r>0$, independent of h, such that

$$\operatorname{supp}\mu_h\subset C(0,r). \tag{5.1}$$

(b) $\forall f\in H^h$,

$$f(x)=\sum_{j\in Z^s}\mu_h f(\cdot+j)\,\psi_h(x-j), \tag{5.2}$$

where $\psi_h=\sigma_{1/h}\phi_h$.

(c) $$|\mu_h f|\leqslant c\,\|f\|_{d,C(0,r)},\qquad \forall f\in C^d(R^s). \tag{5.3}$$

For convenience, we call μ_h a co-quasi-interpolation functional (for Φ) and denote $\mu_h\in M_Q^h:=\Lambda_Q^h\sigma_h$. By (4.3), we can characterize M_Q^h as follows. $\forall\mu_h\in M_Q^h$,

$$\mu_h g^h=b_h(T_{\psi_h}^1)\,g^h(0)=[\delta_0 b_h(T_{\psi_h}^1)]g^h,\qquad \forall g\in H.$$

By (3.9) and (3.11) (recall that Φ is a G-family), we see that the functionals $\delta_0 b_h(T_{\psi_h}^1)$ are uniformly continuous on $C_0(R^s)$ with respect to $h>0$. Since μ_h satisfies (5.1), we can choose a fixed set of functionals $\{\lambda_j\}_{j=1}^m$, such that $\operatorname{supp}\lambda_j\subset H^*$, in order to obtain the representation

$$\mu_h=\sum_{j=1}^m a_j^h\,\lambda_j.$$

Thus, the property of μ_h is mainly determined by $\{a_j^h\}$. These types of functionals are called admissible, first introduced in [10] for the study of quasi-interpolation functionals on the spaces spanned by integer-translates of box-splines.

In this section, we will restrict our attention to functionals λ of differential type, defined by

$$\lambda=\sum_{i\in I}\delta_{y_i}P_i(D), \tag{5.4}$$

where $y_i\in R^s$, $P_i\in\pi$ and I is a finite set. Let

$$P_i(x)=\sum_{0\leqslant|j|\leqslant d_i}c_{ij}x^j.$$

Then

$$\lambda=\sum_{i\in I}\delta_{y_i}\sum_{0\leqslant|j|\leqslant d_i}c_{ij}D^j.$$

So, be setting

$$|||\lambda||| = \max\{|c_{ij}|, 0 \leqslant |j| < d_i, i \in I\},$$

and

$$d = \deg \lambda = \max_{i \in I} \deg P_i,$$

we have, $\forall f \in C^d(R^s)$,

$$|\lambda f| \leqslant c\ |||\lambda|||\ \|f\|_{d, C(0,r)}, \tag{5.5}$$

where $y_i \in C(0, r)$, $i \in I$, and c is a constant independent of $\{c_{ij}\}$.

Later the set of all functionals of differential type will be denoted by Γ.

DEFINITION 5.1. Let H be a d-dimensional TIS and Φ a G-family such that $[H|\Phi]_h = 0$. A linear independent set $\Lambda = \{\lambda_j\}_{j=1}^d \subset \Gamma$ is called an admissible set (with respect to H for Φ) if there exists a co-quasi-interpolation functional $\mu_h \in \operatorname{span} \Lambda$.

It is obvious that within the span of Λ, the co-quasi-interpolation functional for Φ is unique. Now we give a sufficient condition for a subset of Γ to be admissible. First, we need a lemma which is a direct consequence of [4, Theorem 1.2].

LEMMA 5.1. *Let H be a TIS. There exists a unique subspace $\pi_H \subset \pi$ with* $\dim \pi_H = \dim H$, *such that each $p \in \pi_H$ is a limit in the $C^\infty(R^s)$ topology of some family $\{f^h : f^h \in H^h\}$, as $h \to 0$.*

Our result on admissible sets can be stated as follows.

THEOREM 5.1. *Let H be a TIS, Φ a G-family such that $[H|\phi]_h = 0$, and $\Lambda \subset \Gamma$ satisfying $\pi_H \in I_d(\Lambda)$. Then Λ is an admissible set with respect to H for Φ.*

Proof. Write $\Lambda = \{\lambda_j\}_{j=1}^d$, where $d = \dim H$. Since $\pi_H \in I_d(\Lambda)$, there is a basis $\{p_j\}_{j=1}^d$ of π_H such that $\lambda_j p_k = \delta_{j,k}$, $1 \leqslant j$, $k \leqslant d$. For any p_k, we choose $\eta_k^h \in H^h$ so that $\eta_k^h \to p_k$ in the $C^\infty(R^s)$ topology as $h \to 0$. Then

$$\lim_{h \to 0} \lambda_j \eta_k^h = \delta_{jk}, \qquad 1 \leqslant j, \quad k \leqslant d. \tag{5.6}$$

Write $M^h = (\lambda_j \eta_k^h)_{1 \leqslant j, k \leqslant d}$. Then by (5.6), we have $\lim_{h \to 0} M^h = I$, so that

$$\|(M^h)^{-1}\| \leqslant c_1, \tag{5.7}$$

for all sufficiently small h > 0, where c_1 is some constant independent of h.

Next, we consider the sequence $\mathbf{s}_\eta^h = \{b_h(T^1_{\psi_h})\,\eta_j^h(0)\}_{j=1}^d$. Recall that $\operatorname{supp}(\delta_0 b_h(T^1_{\psi_h})) \subset C(0, r)$ for some $r > 0$ independent of h, and $\{\delta_0 b_h(T^1_{\psi_h})\}$ is uniformly bounded on $C_0(R^s)$. Hence there exists a constant $c_2 > 0$, such that

$$\sum_{j=1}^d |b_h(T^1_{\psi_h})\,\eta_j^h(0)| \leqslant c_2,$$

and consequently

$$\sum_{j=1}^d |b_h(T^1_{\psi_h})\,p_j(0)| \leqslant c_2,$$

for all sufficiently small $h > 0$.

Now set $\mathbf{a}^h = (M^h)^{-1}\,\mathbf{s}_\eta^h$ and $\mu_h = \sum_{j=1}^d a_j^h \lambda_j$. Then

$$|||\mu_h||| \leqslant \max_{1 \leqslant j \leqslant d} |||\lambda_j||| \sum_{j=1}^d |a_j^h| \leqslant c_1 c_2 \max_{1 \leqslant j < d} |||\lambda_j|||$$

and

$$\mu_h \eta_j^h = \sum_{k=1}^d a_k^h \lambda_k \eta_j^h = b_h(T^1_{\psi_h})\,\eta_j(0).$$

This means that μ_h is a co-quasi-interpolation functional for Φ.

Theorem 2.1 tells us that if $H \in I(\pi_{n-1}^*)$, then a quasi-interpolation can be used to achieve the approximation order n. A particular (and useful) case is $H \in I_d(\pi_{n-1}^*)$ (i.e., $\dim H = \dim \pi_{n-1}$). In this case it is easy to verify that $\pi_H = \pi_{n-1}$.

COROLLARY 5.1. *Let $H \in I_d(\pi_{n-1}^*)$, Φ a G-family with $[H \mid \Phi]_h = 0$, and $\Lambda \subset \Gamma$ satisfying $\pi_{n-1} \in I_d(\Lambda)$. Then Λ is an admissible set.*

In general, the limit space π_H of H is not easy to find. But if $\psi_h = \sigma_{1/h}\phi_h$ is uniformly convergent to some function $\psi(\cdot)$, as $h \to 0$, i.e.,

$$\lim_{h \to 0} \|\psi_h - \psi\| = 0,$$

then $\operatorname{supp} \psi \subset C(0, r)$, and $\hat{\psi}_h \to \hat{\psi}$ in the $\mathscr{E}$-topology. This, in turn, means that $\hat{\psi}_h$ is uniformly convergent to $\hat{\psi}$ in any compact set in $\mathbb{C}^s$ as $h \to 0$. It therefore follows, by applying (3.2), that

$$\begin{cases} \hat{\psi}(0) \neq 0, \\ p(-iD)\,\hat{\psi}(2\pi j) = 0, \qquad \forall p \in \pi_H, \quad j \in Z^s \setminus \{0\}. \end{cases}$$

Hence, $\pi_H = \pi \cap S(\psi) := \pi_\psi$, where

$$S(\psi) = \operatorname{span}\{\psi(\cdot - j);\, j \in Z^s\}.$$

This means that $S_h(\phi_h)$ and $S(\psi)$, as well as the quasi-interpolation functionals for Φ and for ψ, are intimately related as follows.

THEOREM 5.2. *Let H be a TIS, Φ a G-family such that $[H \mid \Phi]_h = 0$, and $\psi_h = \sigma_{1/h}\phi_h$ be uniformly convergent. Also, let Λ be an admissible set with respect to H for Φ. Then the co-quasi-interpolating functionals $\mu_h \in \operatorname{span} \Lambda$ are convergent in the norm $|||\cdot|||$, as $h \to 0$, and their limit μ is a quasi-interpolation function for ψ.*

Proof. Let $\pi_\psi = \pi \cap S(\psi)$. Then $\dim \pi_\psi = \dim H$. Write $\Lambda = \{\lambda_j\}_{j=1}^d$. As in Theorem 5.1, we choose $\{p_j\}_{j=1}^d \subset \pi_\psi$ and $\{\eta_j^k\}_{j=1}^d \subset H^h$. Since $\eta_j^h \to p_j$ in the $C^\infty(R^s)$ topology and $\psi_h \to \psi$ uniformly as $h \to 0$, we obtain

$$\lim_{h \to 0} b_h(T_{\psi_h}^1)\, \eta_j^h(0) = \lim_{h \to 0} b(T_\psi^1)\, p_j(0),$$

where $b(x) = x^{-1}(1 - (1 - x/\hat{\psi}(0))^d)$ and $d = \dim H$.

As in Theorem 5.1, we set $\mathbf{a}^h = (M^h)^{-1}\, \mathbf{s}_\eta^h$. Then

$$\lim_{h \to 0} \mathbf{a}^h = \lim_{h \to 0} (M^h)^{-1}\, \mathbf{s}_\eta^h = \{b(T_\psi^1) p_j(0)\}_{j=1}^d.$$

Defining

$$\mu = \sum_{j=1}^d \lambda_j b(T_\psi^1) p_j(0),$$

which is obviously a quasi-interpolation functional for ψ, we obtain

$$\lim_{h \to 0} |||\mu_h - \mu||| = 0.$$

As an application of Theorem 5.2, we discuss the dual basis of integer translates of an exponential box spline. Let $X = \{x_1, ..., x_n\} \subset R^s \setminus \{0\}$, $\mu = \{\mu_1, ..., \mu_n) \in \mathbb{C}^n$, and $X_\mu = \{(x_j, \mu_j) \mid x_j \in X, \mu_j \in \mathbb{C}\}$. Then the exponential box spline $B_h(\cdot \mid X_\mu)$ is defined to be the linear functional on $L_{\text{loc}}^1(R^s)$, namely,

$$\phi \mid \to h^{s-n} \int_{[0,h]^n} e^{-\mu \cdot u} \phi\left(\sum_{i=1}^n x^j u_j\right) du.$$

When $\mu=0$, $h=1$, we have the polynomial box spline $B(\cdot\,|\,X)$. For simplicity, we will later write $\phi_h=B_h(\cdot\,|\,X_\mu)$ and $\psi=B(\cdot\,|\,X)$. If

$$\operatorname{span}(X\backslash\{x_i\})=R^s, \qquad 1\leqslant i\leqslant n, \tag{5.8}$$

then $\phi_h\in C_0(R^s)$ and $\{\phi_h\}_{h>0}$ forms a G-family.

Remark. When we study exponential box splines, we usually assume that span $X=R^s$, in order that ϕ_h is a regular function. Under this assumption, if (5.8) does not hold, then ϕ_h can be studied by direct calculation or regularization (cf. [18]). Hence, in this section we always assume that (5.8) holds.

Under the condition (5.8), it is known that

$$\lim_{h\to 0}\|\phi_h(h\cdot)-\psi(\cdot)\|=0. \tag{5.9}$$

Now we turn to the discussion of the dual basis for the exponential box splines.

DEFINITION 5.2. A system of functionals $\{\lambda_j^h\}_{j\in hZ^s}$ is called a dual basis of $\{\phi_h(\cdot-j)\}_{j\in hZ^s}$, if it satisfies

$$\lambda_j^h\phi_h(\cdot-k)=\delta_{j,k}, \qquad \forall j,k\in hZ^s.$$

To construct a dual basis, we first select a functional λ^h that satisfies

$$\lambda^h\phi_h(\cdot-j)=\delta_{0,j}, \tag{5.10}$$

and set $\lambda_j^h=\lambda^h\tau_j$, $j\in hZ^s$. Then $\{\lambda_j^h\}$ is indeed a dual basis of $\{\phi_h(\cdot-j)\}$. We shal call λ^h a dual functional.

Recall from (4.4), that

$$\sum_{j\in hZ^s} g(y+j)\,\phi_h(x-z)=\sum_{j\in hZ^s} g(x+j)\,\phi_h(y-j), \qquad \forall g\in H.$$

Thus, it follows that a dual functional for ϕ_h is also an H-reproducing functional. We recall the following result from [15, 17].

THEOREM C. *Suppose that ϕ_h satisfies the following conditions:*

$$\text{(1)}\quad \hat{\phi}_h(-i\theta/h)\neq 0, \qquad \theta\in\Theta, \tag{5.11}$$

where Θ is the eigenvalue set of H, and

$$\text{(2)}\quad X\subset Z^s\backslash\{0\} \text{ is unimodular.} \tag{5.12}$$

Let π_H be the limit space of H as in Lemma 5.1. Then if for any $t \in \text{Int}[x]$, there exists a unique polynomial $q_t^h \in \pi_H$ such that the functionals λ_j^h, given by $\lambda_j^h := \delta_{t+j} q_t^h(D) \sigma_{1/h}$, satisfy $\lambda_j^h \phi_h(\cdot - k) = \delta_{j,k}$, $\forall j, k \in hZ^s$.

As pointed out in [4], (5.11) automatically holds for all sufficiently small $h > 0$, and $\lambda^h := \delta_t q_t(D) \sigma_{1/h}$ is an H-reproducing functional for ϕ_h as long as (5.12) holds.

However, an H-reproducing functional needs not be a quasi-interpolation functional, unless the uniformly K-bounded condition (2.5) holds. By using Theorem 5.2, we obtain

PROPOSITION 5.1. *Suppose that* (5.12) *holds and $q_t \in \pi_H$ is chosen as in Theorem C. Then the functional $\mu^h = \delta_t q_t^h(D)$ is uniformly bounded, namely,*

$$|||\mu^h||| \leqslant C.$$

Furthermore, $q_t^h \to q_t \in \pi_H$ in the C^∞ topology as $h \to 0$, and

$$\delta_t q_t(D) \psi(\cdot - k) = \delta_{0k}, \qquad \forall k \in Z^s. \tag{5.13}$$

Proof. Let $\{p_j\}_{j=1}^d$ be a basis of π_H. Then for any $t \in \text{Int}[X]$, we have

$$\delta_t p_j(D)\{p_k(\cdot - t)\} = \delta_{kj}, \qquad 1 \leqslant k, \quad j \leqslant d.$$

By Theorem 5.1, $\Lambda = \{\delta_t p_j(D)\}_{j=1}^d$ is an admissible set with respect to H for Φ, and thus there exists a unique $\mu_h \in \text{span}\, \Lambda$ which is a co-quasi-interpolation functional for ϕ. It is easy to check that there is a unique H^h-reproducing functional in span Λ. Since $\delta_t q_t^h(D)$ is a H^h-reproducing functional in span Λ, it must coincide with μ_h. The rest is a consequence of Theorem 5.2 and the fact that if X is unimodular, $q_t \in \pi_\psi$ with $\delta_t q_t(D)$ being a π_ψ-reproducing functional, then (5.13) holds (cf. [17]).

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